LECTURE 9: FERMIONS AND THE DIRAC OPERATOR

Last week we have seen that to describe fermions quantum-mechanically, one needs the theory of *Grassmann variables*. This week we aim for the classical mechanical description of fermions, which leads us to the theory of spinors and the Dirac equation. The associated differential operator that appears in this equation is called the *Dirac operator*, and has become increasingly important also in Mathematics, most notably geometry and topology, since the pioneering work of Atiyah and Singer in the 60's and 70's.

1. Motivation: the Dirac equation

We start this lecture by a more physical introduction. The Dirac operator was invented in 1928 by P.A.M. Dirac as part of his theory of spin 1/2 elementary particles such as the electron. Here we briefly recall the main points of his argument deriving what is now called the *Dirac equation*, as it already highlights some of the main characteristics of the general theory.

We work in Minkovski spacetime which is just $\mathbb{R}^4$ equipped with the $-+++$ metric $\eta_{\mu\nu}$. As usual, we write the coordinates of a point in spacetime as $x^\mu := (t,x,y,z)$, where t is the time coordinate and (x,y,z) the space coordinates. As explained in Lecture II, the Laplacian in this Minkovski spacetime is given by

$$\Box := \sum_{\mu,\nu} \eta^{\mu\nu} \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{1}$$

This operator is also called *the d'Alambertian operator*. The fundamental equation of motion for a scalar (i.e., spin 0) field $\phi(x)$ is given by the so-called *Klein–Gordon* equation

$$(\Box + m^2)\phi = 0,$$

which expresses the fundamental relativistic relation between energy and mass m. For spin 1/2 particles, Dirac was looking for a first order field equation, compatible with the Klein–Gordon equation. For this he postulated a general first oder equation of the form

$$(D - m)\psi = 0, \tag{2}$$

where $D = \sum_\mu \gamma^\mu \frac{\partial}{\partial x^\mu}$ is a general first order differential operator in all variables, with coefficients γ^μ not depending on the coordinates x^μ. Suppose that the field $\psi(x)$ satisfies

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equation (**??**). Applying the operator D once more to this equation, we get

$$0 = (D^2 - m^2)\psi$$

which must be equal to (minus) the Klein–Gordon equation in order not to violate the special theory of relativity. Therefore we must have

$$-\Box = D^2 = \frac{1}{2}\sum_{\mu,\nu}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu},$$

which leads to the requirement

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu} \tag{3}$$

Clearly the γ^μ cannot be ordinary numbers in order to satisfy these relations, but Dirac found the following 4×4-matrices that do satisfy these relations:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},\ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},\ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

These matrices are the basic ingredients of Dirac's theory: Spin $1/2$ particles in field theory are not described by scalar fields, but rather $\psi(x)$ is vector-valued function with values in $\mathbb{C}^4$. The classical field equation satisfied by this field is the *Dirac equation* (**??**). The operator D appearing in this equation is called the *Dirac operator*.

Recall that the fundamental property of the Klein–Gordon equation was its invariance under the Lorentz group. So what about the Dirac equation? The Lorentz group is simply $SO(3,1)$, the group of all linear maps preserving the metric $\eta_{\mu\nu}$. This is a Lie group whose Lie algebra is spanned by $J_a,\ a = 1,2,3$ (generating spatial rotations) and $K_a,\ a = 1,2,3$ (generators of boosts) satisfying

$$\begin{aligned}[K_a, K_b] &= -i\epsilon^{abc}J_c \\ [J_a, K_b] &= i\epsilon^{abc}K_c \\ [J_a, J_b] &= i\epsilon^{abc}J_c\end{aligned}$$

Defining $\xi_a^\pm := \frac{1}{2}(J_a \pm iK_a)$, these relations transform to

$$[\xi_a^\pm, \xi_b^\pm] = i\epsilon^{abc}\xi_c^\pm, \qquad [\xi_a^+, \xi_b^-] = 0. \tag{4}$$

Of course the well-known Pauli spin matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfy very similar commutation relations, namely

$$\left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2}\right] = i\epsilon^{abc}\frac{\sigma_c}{2}.$$

We can also recognize these matrices as blocks in the γ-matrices, and this suggests the following action of the Lie algebra of the Lorentz group on spinors: we decompose $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$, with the Lie algebra (**??**) acting on the first copy of $\mathbb{C}^2$ sending $\xi_a^+ \mapsto \sigma_a/2$ and $\xi_a^- \mapsto 0$. For the second copy we have the opposite: $\xi_a^- \mapsto \sigma_a/2$ and $\xi_a^+ \mapsto 0$. Upon exponentiating, we find that a Lorentz transformation with rotation angles $\vec{\theta}$ and boost parameters $\vec{\phi}$ is implemented by $g(\vec{\theta},\vec{\phi}) \in GL(4,\mathbb{C})$ defined by

$$g(\vec{\theta},\vec{\phi}) := \begin{pmatrix} \exp\left(\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\phi})\right) & 0 \\ 0 & \exp\left(\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\phi})\right) \end{pmatrix} \tag{5}$$

With this transformation rule, one can check that the Dirac equation is indeed invariant. The projector onto each of the two copies of $\mathbb{C}^2$, which are invariant subspaces of this representation, is given by

$$\pi^{\pm} = \frac{1}{2}(1 \pm \gamma^5), \quad \gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3.$$

Also remark that with respect to this decomposition $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$ the Dirac operator maps the two subspaces to each other and takes the form

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

However, an important observation is that the representation (**??**) is not quite a representation of the Lorentz group! Indeed if we just consider a rotation around the z-axis with angle $\alpha \in [0,2\pi)$, this implemented by the matrix

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}.$$

But this equals minus the identity for $\alpha = 2\pi$ and only return to the identity at $\alpha = 4\pi$. What is true is that the representation (**??**) is a representation of a *double cover* of the Lorentz group, which happens to be the group $SL(2,\mathbb{C})$. This means that there is a covering map from $SL(2,\mathbb{C})$ to the Lorentz group with kernel equal to $\mathbb{Z}/2\mathbb{Z}$:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SL(2,\mathbb{C}) \longrightarrow SO(3,1) \longrightarrow 1. \tag{6}$$

Conclusions. The theory above gives a description of fermions on MInkovski space, its basic ingredient being the *Dirac operator* $D = \sum_\mu \gamma^\mu \partial/\partial x^\mu$. Before we turn to problem of generalizing this Dirac operator to general manifolds, we draw some conclusions from the preceding discussion:

i) *Fermions are vector valued rather than scalar fields*: over a general manifold they will be given by sections of a vector bundle $S \to M$.

ii) *We need matrices satisfying the commutation relations* (**??**). Mathematically this is done by the theory of Clifford algebras and their representations. The crucial difference is that on a general manifold the resulting "gamma-matrices" $\psi^i(x)$

will depend on the basepoint $x \in M$. To give away the punchline, the Dirac operator will look like

$$D = \sum_{i=1}^{n} \psi^i(x) \nabla^S_{\partial/\partial x^i},$$

in local coordinated $x = (x^1, \ldots, x^n)$ on M. Here ∇^S is a suitable connection on the bundle S.

iii) *When it comes to invariance, we should be prepared to accept double coverings of symmetry groups*. This is a theme that will be picked up in a later lecture.

2. Clifford Algebras

As it turned out, the relations (**??**) had been considered from an abstract point of view by the english mathematician William Clifford in 1876 in an attempt to generalize Hamiltons' quaternions to higher dimensions. The upshot of the story is that the equations (**??**) abstractly define an *algebra* (i.e., a vector space equipped with an associative multiplication), and the Dirac γ-matrices define a specific representation of this algebra, called the *spinor representation*. From the abstract point of view, there is really no need to restrict to the case of Minkovski space, it works just as well for any inner product space:

Definition 2.1. Let (V, η) be a vector space V equipped with an inner product η. The *Clifford algebra* $\mathrm{Cliff}(V, \eta)$ is the algebra generated by a linear map $v \mapsto \psi(v)$, $v \in V$, subject to the relations

$$\psi(v_1)\psi(v_2) + \psi(v_2)\psi(v_1) = -2\eta(v_1, v_2), \quad \text{for all } v_1, v_2 \in V. \tag{7}$$

Remark 2.2. There is actually no need for the inner product η to be non-degenerate, we can even put it to zero! In that case ($\eta = 0$) the Clifford relations amount to the fact that the $\psi(v)$ all anti-commute with each other, and therefore we see that $\mathrm{Cliff}(V, 0) = \bigwedge V$, the Grassmann algebra. Choosing a basis $(e_1, \ldots, e_n)$ for V, and defining $\theta_i := \psi(e_i)$, $i = 1, \ldots, n$, we get exactly the algebra of Grassmann numbers (or variables) considered in the previous lecture.

For a general inner product space (V, η) we can choose an orthonormal $(e_1, \ldots, e_n)$ so that $V \cong \mathbb{R}^n$ with inner product given by

$$\eta(x, y) = x^1 y^1 + \ldots + x^p y^p - x^{p+1} y^{p+1} - \ldots - x^n y^n, \tag{8}$$

Putting $q := n - p$, we denote this space by $\mathbb{R}^{p,q}$, so that In this notation $\mathbb{R}^{3,1}$ is good old Minkovski space. We write $\mathrm{Cliff}_{p,q}$ for the Clifford algebra of $\mathbb{R}^{p,q}$. In terms of the basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^{p,q}$ we define $\psi_i := \psi(e_i)$. Then we see that a basis for $\mathrm{Cliff}_{p,q}$ is given by

$$\{1, \psi_{i_1} \cdots \psi_{i_k} \,|\, i_1 < i_2 < \ldots < i_k,\ k = 1, \ldots, p + q\}, \tag{9}$$

and therefore its dimension is 2^n. The multiplication between the basis elements is given by

$$\psi_i\psi_j = -\psi_j\psi_i,\ i \neq j, \qquad \psi_i^2 = \begin{cases} -1 & 1 \leq i \leq p \\ 1 & p+1 \leq i \leq p+q. \end{cases} \tag{10}$$

This gives a very concrete description of the Clifford algebra.

Example 2.3. Let us work out some easy examples for low degrees of p and q

i) For $p = 0 = q$ we have $\mathrm{Cliff}_{0,0} = \mathbb{R}$. (This is more a definition.)

ii) For $p = 1, q = 0$, $\mathrm{Cliff}_{1,0}$ is 2-dimensional, and has one generator ψ satisfying $\psi^2 = -1$. This is just the complex numbers: $\mathrm{Cliff}_{0,1} = \mathbb{C}$!

iii) The algebra $\mathrm{Cliff}_{2,0}$ is 4-dimensional and has generators 1, $\mathbf{i} := \psi_1$, $\mathbf{j} := \psi_2$ and $\mathbf{k} := \psi_1\psi_2$. Working out the relations we see:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,\ \mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{ijk} = -1, \text{ etc.}$$

These are the quaternions $\mathbb{H}$.

iv) The algebra $\mathrm{Cliff}_{0,1}$ is 2-dimensional with generator ψ satisfying $\psi^2 = 1$. This can only be $\mathbb{R} \oplus \mathbb{R}$, equipped with the component-wise multiplication.

v) The algebra $\mathrm{Cliff}_{0,2}$ is 4-dimensional. The map defined on generators by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \psi_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

gives and isomorphism $\mathrm{Cliff}_{0,2} \cong M_2(\mathbb{R})$.

vi) Very relevant to physics is of course the algebra $\mathrm{Cliff}_{3,1}$ which corresponds to Minkovski space $\mathbb{R}^{3,1}$. In this case, as explained in previous section, the Dirac's γ-matrices satisfy the Clifford relations (**??**). Mathematically, this means that they define a *representation* of $\mathrm{Cliff}_{3,1}$. In the next section we shall study representations of Clifford algebras in general.

Definition 2.4. The *volume element* in $\mathrm{Cliff}_{p,q}$ is defined by

$$\tau := \psi_1 \cdots \psi_n.$$

Lemma 2.5. *The volume element τ satisfies*

$$\tau^2 = (-1)^{\frac{n(n-1)}{2}+p}, \quad \psi_i\tau = (-1)^{n-1}\tau\psi_i$$

The proof of this Lemma is an exercise.

3. Representations of Clifford algebras

In mathematics, we usually distinguish between abstract algebraic structures and relations (such as the algebra $\mathrm{Cliff}_{3,1}$) and concrete incarnations on vector spaces (such as the Dirac γ-matrices. For example, an abstract group G is a set equipped with an associative multiplication which has a unit and inverses. A representation of G is given by a vector space V together with a homomorphism $\varphi : G \to GL(V)$. Concretely, this means that we can represent each element $g \in G$ by an invertible matrix $\varphi(g)$ so that the matrices satisfy all the relations defining the specific group G. The mathematical study of all representations of a given group is called its *representation theory*. This fits nicely with the paradigm of Quantum Mechanics: classical mechanics is usually described in terms of a manifold with *symmetries* given by a group G acting on it. The associated Quantum Mechanical system is as usual formulated in terms of a Hilbert space $\mathcal{H}$ (in particular a vector space), where the symmetries are represented by invertible linear operators, i.e., a representation of G!

Coming back to Clifford algebras, the situation is similar.

Definition 3.1. A *representation* of $\mathrm{Cliff}(V,\eta)$ is given by a complex vector space M together with a homomorphism of algebra $\mathrm{Cliff}(V,\eta) \to \mathrm{End}(M)$.

Example 3.2. Consider the exterior algebra $\bigwedge V$ of V. Given $v \in V$, we consider the following two operators on $\bigwedge V$: first we have $\varepsilon(v)$, given by exterior multiplication by v, and second $\iota(v)$ given by contraction with the covector $\eta(v,-) \in V^*$:

$$\begin{aligned}\varepsilon(v)(v_1 \wedge \ldots \wedge v_k) &:= v \wedge v_1 \wedge \ldots \wedge v_k,\\ \iota(v)(v_1 \wedge \ldots \wedge v_k) &:= \sum_i (-1)^i \eta(v,v_i) v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_k,\end{aligned}$$

where the hat means omission from the argument. With these two operators, the combination

$$\psi(v) := \varepsilon(v) - \iota(v) \in \mathrm{End}(\textstyle\bigwedge V)$$

satisfy the commutation relations (**??**), in other words, define a representation of $\mathrm{Cliff}(V,\eta)$. The only point is perhaps that this is a real vector space. But we can always complexify and consider $\bigwedge V \otimes \mathbb{C}$ to get a complex representation.

4. The Dirac operator

We now return to geometry. Our goal is to define a Dirac operator on a pseudo-riemannian manifold (M,g), generalizing the Dirac operator in Minkovski space discussed in §**??**. In fact we already have all the ingredients for the Dirac operator on flat space, not just Minkovski space. Let (V,η) be a vector space with inner product, and suppose that we are given a representation M of $\mathrm{Cliff}(V,\eta)$. The Dirac operator is a first order differential operator acting on M-valued functions:

$$D = \sum_{i=1}^{n} \psi(e_i) \frac{\partial}{\partial x^i},$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of V. The Clifford relations give the fundamental property

$$D^2 = -\Delta$$

From the formula of the Dirac operator above it becomes immediately clear what is needed to define the Dirac operator on a general manifold: there is no invariant/canonical way to differentiate sections of a vector bundle, for this we have to choose a connection. Therefore, before we proceed, we need to recall the Levi–Civita connection on a a pseudo-riemannian manifold.

4.1. **The Levi–Civita connection.** Let M be a smooth n-dimensional manifold. A *pseudo-riemannian metric* on M is given by an inner product $g_x : T_xM \times T_xM \to \mathbb{R}$ which depends smoothly on $x \in M$. More precisely, it is given by a smooth symmetric tensor field in $T^*M \otimes T^*M$ which defines a indefinite inner product on each tangent space T_xM. In local coordinates $(x^1, \dots x^n) : U \to \mathbb{R}^n$ we can write

$$g(x) = g_{ij}(x) dx^i \otimes dx^j.$$

Locally, we can find an orthonormal frame $\{e_i(x)\}_{i=1}^n$, $x \in U \subset M$ of TM, which brings the metric into normal form (**??**). The pair of integers (p, q) are an invariant of the manifold, called the *signature*. When $q = 0$ we have a riemannian manifold (this is what is usually considered in mathematics, here the metric is positive definite), and when $p = 3, q = 1$ we say the manifold is *Lorentzian*.

Proposition 4.1. *Given a riemannian metric g, there exists a unique connection on TM, called the* Levi–Civita connection ∇^{LC} *satisfying*

$i)$ *(compatibility with the metric)*

$$Zg(X,Y) = g(\nabla^{\mathrm{LC}}_Z X, Y) + g(X, \nabla^{\mathrm{LC}}_Z Y), \quad \textit{for all vector fields } X, Y, Z \in \mathfrak{X}(M),$$

$ii)$ *(Torsion-free)*

$$\nabla^{\mathrm{LC}}_X Y - \nabla^{\mathrm{LC}}_Y X = [X, Y].$$

This connection is uniquely determined by the equation

$$g(\nabla_X Y, Z) = \frac{1}{2} \left(Xg(Y,Z) - Zg(X,Y) + Yg(Z,X) \right.$$
$$\left. - g(X,[Y,Z]) + g(Z,[X,Y]) + g(Y,[Z,X]) \right). \tag{11}$$

Proof (sketch): First check that properties $i)$ and $ii)$ imply (**??**). Then prove that the right hand side, for fixed X, Y is tensorial in Z in the sense that it is $C^\infty(M)$-linear. Conclude that we can write $\omega(Z)$ for the right hand side, for a unique one-form ω. Using the nondegeneracy of g, define $g(\nabla^{\mathrm{LC}}_X Y, Z) = \omega(Z)$. Then check that the ∇^{LC} thus defined is indeed a connection. Finally, the metric and torsion free property follow from (**??**). This same equation also shows that ∇ is unique. $\square$

In local coordinates the connection writes out as

$$\nabla_{\partial/\partial x^i} \sum_j X^j \frac{\partial}{\partial x^j} = \sum_j \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j} + \sum_{jk} \Gamma^j_{ik} X^k \frac{\partial}{\partial x^j},$$

where Γ^j_{ik} are called the *Christoffel symbols*. This corresponds to the usual decomposition $\nabla^{\mathrm{LC}} = d + \Gamma$ in the local trivialization of TM induced by the choice of local coordinates, with $\Gamma \in \Omega^1(M, \mathrm{End}(TM))$. We see from equation (**??**) that the Christoffel symbols are defined by the metric by the formula

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

where g^{ij} denotes the components of the inverse of the matrix (g_{ij}).

4.2. **Clifford modules.** Let (M, g) be a pseudo-riemannian manifold of signature (p, q). For each point $x \in M$, the tangent space T_xM comes equipped with the metric $g_x : T_xM \times T_xM \to \mathbb{R}$, and we can consider the Clifford algebra $\mathrm{Cliff}(T_xM, g_x)$ associated to this metric space. Varying the basepoint $x \in M$, we obtain in this way a bundle of Clifford algebras $\mathrm{Cliff}(TM, g)$, with the property that for each $x \in M$ there is an isomorphism $\mathrm{Cliff}(T_xM, g_x) \cong \mathrm{Cliff}_{p,q}$.

Definition 4.2. A *Clifford bundle* over (M, g) is a vector bundle $S \to M$ with connection ∇^S with the following properties:

i) for each $x \in M$, S_x is a representation of $\mathrm{Cliff}(T_xM, g_x)$, i.e., S is a bundle of Clifford representations,

ii) the connection ∇^S is compatible with the Levi-Civita connection in the following way:

$$\nabla^S_X(Y \cdot s) = (\nabla^{\mathrm{LC}}_X Y) \cdot s + Y \cdot (\nabla^S_X s),$$

for all $X, Y \in \mathfrak{X}(M)$ and s a section of S. In this equation, the $\cdot$ indicates the Clifford action of a vector field, i.e., a section of $TM \subset \mathrm{Cliff}(TM, g)$ on S.

Example 4.3. The easiest example of a Clifford bundle is simply the bundle version of the easiest example **??** of a representation of the Clifford algebra. However, now it is more natural to consider the dual $\bigwedge T^*M$ because sections of this bundle are differential forms. Therefore we define $\iota(X)$ to be the contraction with a vector field $X \in \mathfrak{X}(M)$, and $\varepsilon(X)$ to be the wedge product with the one form $g(X, -) \in \Omega^1(M)$ dual to X, and define

$$\psi(X) := \epsilon(X) - \iota(X).$$

As for the case that M equals a point in Example **??**, this turns $\bigwedge T^*M$ into a bundle of representations of $\mathrm{Cliff}(TM)$. As we have seen in Remark **??**, the Levi–Civita connection induces a connection on $\bigwedge T^*M = \bigoplus_{k\geq 0} \bigwedge^k T^*M$, and one can check that with this connection, the bundle becomes a Clifford bundle.

4.3. **The Dirac operator associated to a Clifford bundle.** The key point is now that a Clifford bundle contains just the right amount of information to define a Dirac operator:

Definition 4.4. Let (S, ∇^S) be a Clifford bundle over a pseudo-riemannian manifold (M, g). The *Dirac operator* D associated to S is the differential operator given by the composition

$$\Gamma(M,S) \xrightarrow{\nabla^S} \Gamma^\infty(M, T^*M \otimes S) \xrightarrow{T^*M \overset{g}{\cong} TM} \Gamma^\infty(M, TM\otimes S) \xrightarrow{\substack{\text{Clifford}\\ \text{multiplication}}} \Gamma^\infty(M,S)$$

Suppose that $\{e_i\}_{i=1}^n$ is a local orthonormal system of vector fields on $U \subset M$. Then the definition of the Dirac operator above amounts to the local formula

$$Ds = \sum_{i=1}^n \psi(e_i)\nabla^S_{e_i} s$$

Example 4.5. In the Example **??** of a Clifford bundle, the associated Dirac operator is given by

$$D = \sum_{i=1}^n \epsilon(e_i)\nabla^{\mathrm{LC}}_{e_i} - \sum_{i=1}^n \iota(e_i)\nabla^{\mathrm{LC}}_{e_i} = d + d^*$$

This last identity is a nontrivial computation that we postpone for now.

4.4. **The square of the Dirac operator.** Finally, let us go back to the original motivation and try to relate the square of the Dirac operator to the Laplacian on a Riemannian manifold. On a general manifold, it is no longer true that the Dirac operator squares to the Laplacian:

$$\begin{aligned} D^2 &= \sum_{i,j=1}^n (\psi(e_i)\nabla^S_{e_i})(\psi(e_j)\nabla^S_{e_j}) \\ &= \sum_{i,j=1} \psi(e_i)\psi(e_j)\nabla^S_{e_i}\nabla^S_{e_j} \\ &= -\sum_{i=1}^n (\nabla^S_{e_i})^2 + \sum_{i<j} \psi(e_i)\psi(e_j)(\nabla^S_{e_i}\nabla^S_{e_j} - \nabla^S_{e_j}\nabla^S_{e_i}). \end{aligned}$$

In this computation we have chosen a synchronous frame $\nabla^S_{e_i} e_j = 0$ at a given point $x_0 \in M$. The nature of the two terms above is quite different: the first term is a second order differential operator resembling the Laplacian, whereas the second term is related to the curvature of ∇^S: this is a zeroth order differential operator. We will see in later lectures that this so-called *Weitzenböck formula* for the square of the Dirac operator shows that D^2 and the Laplacian have the same *principal symbol*.